

The free group in R: introducing the freegroup package

Robin K. S. Hankin

Abstract

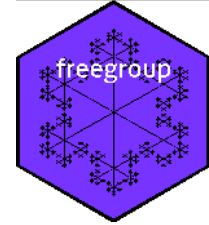
Here I present the **freegroup** package for working with the free group on a finite set of symbols. The package is vectorised; internally it uses an efficient matrix-based representation for free group objects but uses a configurable print method. A range of R-centric functionality is provided. It is available on CRAN at <https://CRAN.R-project.org/package=freegroup>. To cite the **freegroup** package, use [Hankin \(2022\)](#).

Keywords: Free group, Tietze form.

This document has not been peer-reviewed and is not an accepted Tragula publication. It is intended as an example of the style and substance appropriate for submission to Tragula.

1. Introduction

The free group is an interesting and instructive mathematical object with a rich structure that illustrates many concepts of elementary group theory. The **freegroup** package provides some functionality for manipulating the free group on a finite list of symbols. Informally, the *free group* $(X, \circ)$ on a set $S = \{a, b, c, \dots, z\}$ is the set X of *words* that are objects like $W = c^{-4}bb^2aa^{-1}ca$, with a group operation of string juxtaposition. Usually one works only with words that are in “reduced form”, which has successive powers of the same symbol combined, so W would be equal to $c^{-4}b^3ca$; see how b appears to the third power and the a term in the middle has vanished. The group operation of juxtaposition is formally indicated by $\circ$, but this is often omitted in algebraic notation; thus, for example $a^2b^{-3}c^2 \circ c^{-2}ba = a^2b^{-3}c^2c^{-2}ba = a^2b^{-2}ba$.



1.1. Formal definition

If X is a set, then a group F is called *the free group on X* if there is a set map $\Psi: X \rightarrow F$, and for any group G and set map $\Phi: X \rightarrow G$, there is a unique homomorphism $\alpha: F \rightarrow G$ such that $\alpha \circ \Psi = \Phi$, that is, the diagram below commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & F \\ & \searrow \Phi & \downarrow \alpha \\ & & G \end{array}$$

It can be shown that F is unique up to group isomorphism; every group is a quotient of a free group.

1.2. Existing work

Computational support for working with the free group is provided as part of a number of algebra systems including [GAP](#), Sage ([The sage Developers 2019](#)), and [sympy](#) ([Meurer et al. 2017](#)) although in those systems the emphasis is on finitely presented groups, not in scope for the **freegroup** package. There are also a number of closed-source proprietary systems which are of no value here.

2. The package in use

In the **freegroup** package, a word is represented by a two-row integer matrix; the top row is the integer representation of the symbol and the second row is the corresponding power. For example, to represent $a^2b^{-3}ac^2a^{-2}$ we would identify a as 1, b as 2, etc and write

```
> (M <- rbind(c(1,2,3,3,1),c(2,-3,2,3,-2)))
```

```
      [,1] [,2] [,3] [,4] [,5]
[1,]    1    2    3    3    1
[2,]    2   -3    2    3   -2
```

(see how negative entries in the second row correspond to negative powers). Then to convert to a more useful form we would have

```
> library("freegroup")
> (x <- free(M))
```

```
[1] a^2.b^-3.c^5.a^-2
```

The representation for R object x is still a two-row matrix, but the print method is active and uses a more visually appealing scheme. The default alphabet used is **letters**. We can coerce strings to free objects:

```
> (y <- as.free("aabbcccc"))
```

```
[1] a^2.b^3.c^4
```

The free group operation is simply juxtaposition, represented here by the plus symbol:

```
> x + y
```

```
[1] a^2.b^-3.c^5.b^3.c^4
```

(see how the a “cancels out” in the juxtaposition).

2.1. Notation

The package generally uses additive notation but also, as an experimental feature, supports multiplicative notation. Thus $x+y == x*y$. One motivation for the use of “+” rather than “*” is that Python uses “+” for appending strings:

```
>>> "a" + "abc"
'aabc'
>>>
```

However, note that the “+” symbol is usually reserved for commutative and associative operations; string juxtaposition is associative.

Multiplication by integers—denoted in **freegroup** idiom by “*”—is also defined. Suppose we want to concatenate 5 copies of x :

```
> x*5

[1] a^2.b^-3.c^5.b^-3.c^5.b^-3.c^5.b^-3.c^5.b^-3.c^5.a^-2
```

This operation is vectorized:

```
> x*(0:3)

[1] 0 a^2.b^-3.c^5.a^-2
[3] a^2.b^-3.c^5.b^-3.c^5.a^-2 a^2.b^-3.c^5.b^-3.c^5.b^-3.c^5.a^-2
```

There are a few methods for creating free objects, for example:

```
> abc(1:9)

[1] a a.b a.b.c a.b.c.d
[5] a.b.c.d.e a.b.c.d.e.f a.b.c.d.e.f.g a.b.c.d.e.f.g.h
[9] a.b.c.d.e.f.g.h.i
```

And we can also generate random free objects:

```
> rfree(10,4)

[1] a.b^-2.d^2 d.a.d^-2 b^3.a^6.c b^-6.d^-1
[5] d^-1.c^-2.b^2.d^4 b^-4.c.d^3.c^-2 d^-3 a^4.b^2
[9] b^-3.a^3.b^-3 a^-4.c^3.a^4
```

Inverses are calculated using unary or binary minus:

```
> (p <- rfree(10,4))
```

```
[1] d^-2.b^-5      b^3.d.a^-4      d.a.b^7      a^-2.b^-3.a.b^4
[5] b.c^2          b^-1.a.d^-3.c^-1 a^-5.b^-4.a^-3 a^2.c^4.b^-4
[9] c^-8.d^2.a^-2  b^4.d^-1.a^4
```

```
> -p
```

```
[1] b^5.d^2      a^4.d^-1.b^-3  b^-7.a^-1.d^-1  b^-4.a^-1.b^3.a^2
[5] c^-2.b^-1    c.d^3.a^-1.b   a^3.b^4.a^5     b^4.c^-4.a^-2
[9] a^2.d^-2.c^8  a^-4.d.b^-4
```

```
> p~p
```

```
[1] 0 0 0 0 0 0 0 0 0 0
```

We can take the “sum” of a vector of free objects simply by juxtaposing the elements:

```
> sum(p)
```

```
[1] d^-2.b^-2.d.a^-4.d.a.b^7.a^-2.b^-3.a.b^5.c^2.b^-1.a.d^-3.c^-1.a^-5.b^-4.a^-1.c^4.b^-4.
```

Powers are defined as per group conjugation: $x^y == y^{-1}xy$ (or, written in additive notation, $-y+x+y$):

```
> p
```

```
[1] d^-2.b^-5      b^3.d.a^-4      d.a.b^7      a^-2.b^-3.a.b^4
[5] b.c^2          b^-1.a.d^-3.c^-1 a^-5.b^-4.a^-3 a^2.c^4.b^-4
[9] c^-8.d^2.a^-2  b^4.d^-1.a^4
```

```
> a <- alpha(26)
```

```
> p^a
```

```
[1] z^-1.d^-2.b^-5.z      z^-1.b^3.d.a^-4.z      z^-1.d.a.b^7.z
[4] z^-1.a^-2.b^-3.a.b^4.z  z^-1.b.c^2.z          z^-1.b^-1.a.d^-3.c^-1.z
[7] z^-1.a^-5.b^-4.a^-3.z  z^-1.a^2.c^4.b^-4.z    z^-1.c^-8.d^2.a^-2.z
[10] z^-1.b^4.d^-1.a^4.z
```

Thus:

```
> sum(p^a) == sum(p)^a
```

```
[1] TRUE
```

The experimental multiplicative notation allows us to have the equivalent of $(xy)^z = x^zy^z$ and $x^{(yz)} = (x^y)^z$:

```
> x <- rfree()
> y <- rfree()
> z <- rfree()
> all((x*y)^z == x^z * y^z)
```

```
[1] TRUE
```

```
> all(x^(y*z) == (x^y)^z)
```

```
[1] TRUE
```

In additive notation these manifest, somewhat unappealingly, as $(x+y)^z == x^z + y^z$ and $x^{(y+z)} = (x^y)^z$. Further, note that the distributive law $x * (y + z) = x * y + x * z$ is now incorrect [we have, again somewhat unappealingly, $x * (y + z) = x * y * z = x + y + z$] but it can be resurrected if we reinterpret addition as (vector) juxtaposition:

```
> all(x * c(y, z) == c(x*y, x*z))
```

```
[1] TRUE
```

```
> all(c(x, y) * z == c(x*z, y*z))
```

```
[1] TRUE
```

There is also a commutator bracket, defined as $[x, y] = x^{-1}y^{-1}xy$ or in package idiom `. [x, y]` = `-x-y+x+y`:

```
> . [p, a]
```

```
[1] b^5.d^2.z^-1.d^-2.b^-5.z
[2] a^4.d^-1.b^-3.z^-1.b^3.d.a^-4.z
[3] b^-7.a^-1.d^-1.z^-1.d.a.b^7.z
[4] b^-4.a^-1.b^3.a^2.z^-1.a^-2.b^-3.a.b^4.z
[5] c^-2.b^-1.z^-1.b.c^2.z
[6] c.d^3.a^-1.b.z^-1.b^-1.a.d^-3.c^-1.z
[7] a^3.b^4.a^5.z^-1.a^-5.b^-4.a^-3.z
[8] b^4.c^-4.a^-2.z^-1.a^2.c^4.b^-4.z
[9] a^2.d^-2.c^8.z^-1.c^-8.d^2.a^-2.z
[10] a^-4.d.b^-4.z^-1.b^4.d^-1.a^4.z
```

If we have more than 26 symbols the print method runs out of letters:

```
> alpha(1:30)
```

```
[1] a b c d e f g h i j k l m n o p q r s t u v w x y
[26] z NA NA NA NA
```

If this is a problem (it might not be: the print method might not be important) it is possible to override the default symbol set:

```
> options(freegroup_symbols = state.abb)
> alpha(1:30)
```

```
[1] AL AK AZ AR CA CO CT DE FL GA HI ID IL IN IA KS KY LA ME MD MA MI MN MS MO
[26] MT NE NV NH NJ
```

3. Conclusions and further work

The **freegroup** package furnishes a consistent and documented suite of reasonably efficient R-centric functionality. Further work might include the finitely presented groups but it is not clear whether this would be consistent with the precepts of R.

References

- GAP (2018). *GAP – Groups, Algorithms, and Programming, Version 4.10.0*. The GAP Group. URL <https://www.gap-system.org>.
- Hankin RKS (2022). “The free group in R.” doi:10.48550/ARXIV.2212.05883.
- Meurer A, *et al.* (2017). “SymPy: symbolic computing in Python.” *PeerJ Computer Science*, **3**, e103. ISSN 2376-5992. doi:10.7717/peerj-cs.103. URL <https://doi.org/10.7717/peerj-cs.103>.
- The sage Developers (2019). *SageMath, the sage mathematics software system (version 8.6)*. URL <https://www.sagemath.org>.

Affiliation:

Robin K. S. Hankin
University of Stirling
Scotland
email: hankin.robin@gmail.com